A method for computing the curvature of a leaf spring by approximation

When attempting to determine the shape taken by a leaf spring such as is commonly used in vertical-axis seismometers it rapidly becomes apparent that the ordinary mathematical approach used to analyze bending, as applied to bending beams, does not work. The problem is that these approaches depend on the assumption that the deflection of the beam is small, with little change of direction along its length. A strongly bent leaf spring changes direction a lot.

The problem we want to solve is:
Given a leaf spring which is initially straight (flat) and firmly mounted at one end, compute the shape it takes when a known force and moment are applied to its free end.

The parameters we know are:

1.The point of attachment of the spring is assumed to be at the origin (0,0)

2. θ_0, the initial direction of the spring as it leaves its point of attachment, as measured from the horizontal

3. E_1I, the stiffness of the spring, which is the corrected modulus of elasticity of the spring material times its section ‘moment of inertia’.

4. Q_N, the bending moment applied to the free end.

5. F, the magnitude of the force applied to the free end.

6. ϕ, the direction of that force, as measured from the horizontal.

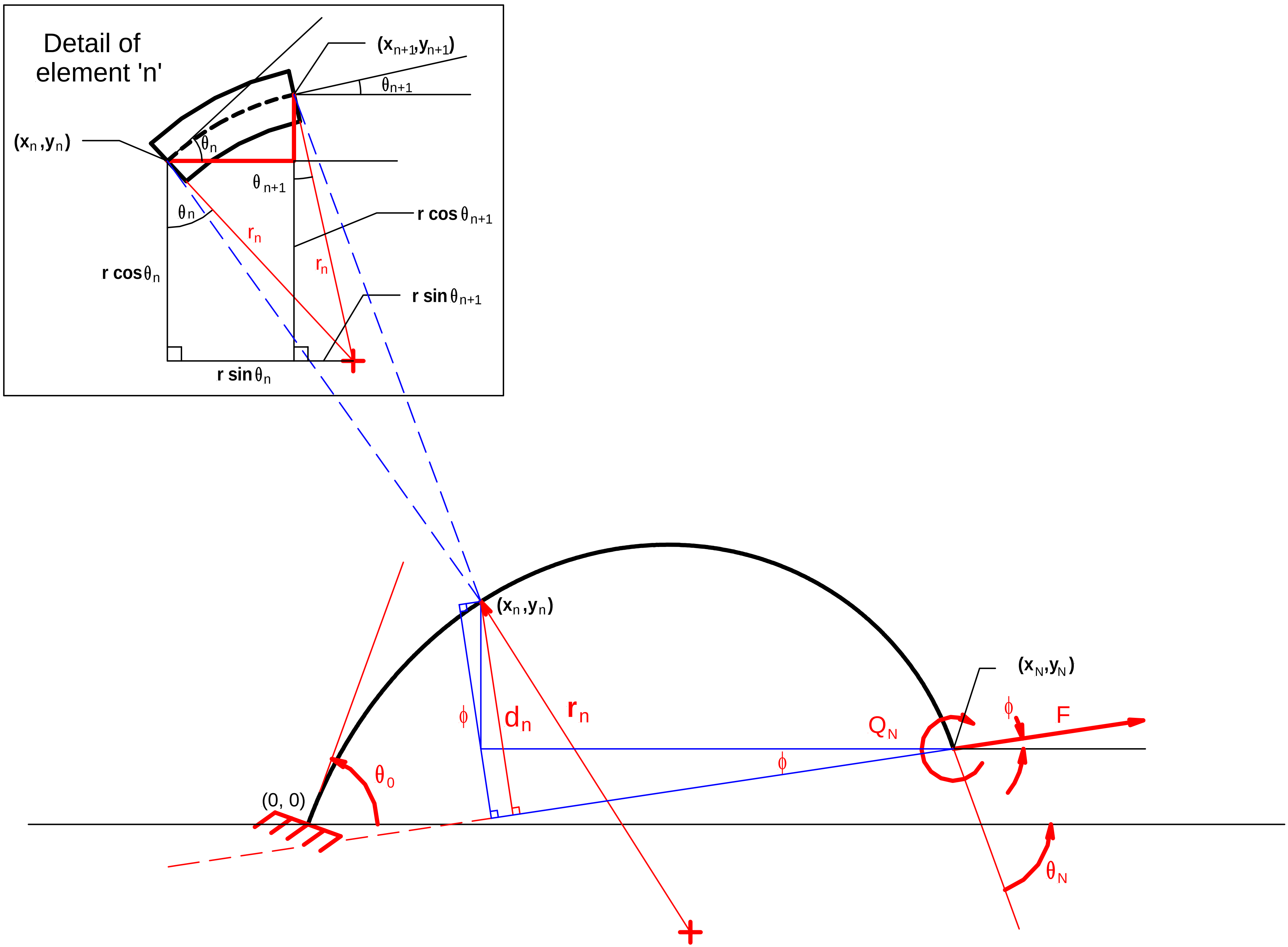
We want to compute:

1. The shape taken by the spring as represented by the x,y coordinates of points along its length, which necessarily will include the coordinates of its free end x_N,y_N.

2. θ_N, the direction of the free end of the spring, as measured from the horizontal.

The approach:

The spring is divided along its length into a large number, N, say 600, extremely small, but still finite elements. The length of each element, s = L / N, where L is the length of the spring. From the location of the first element, we will calculate the location of the second, and using that, the location of the third, and so on until the end of the spring is reached. This is an exact process except for one approximation. The bending radius of each element is assumed not to change within itself, however, a unique bending radius is computed for each element. This approximation seems reasonable, since the bending radius changes quite slowly as one traverses the spring, and since the elements are so small, the change in radius from one element to the next should also be extremely small.


Detail of element 'n'
(x_{n+1},y_{n+1})
θ_{n+1}
(x_n,y_n)
θ_n
θ_{n+1}
θ_n
r_n
$r\cos\theta_{n+1}$
r_n
$r\cos\theta_n$
$r\sin\theta_{n+1}$
$r\sin\theta_n$
(x_n,y_n)
(x_N,y_N)
ϕ
d_n
r_n
Q_N
ϕ
F
ϕ
θ_0
(0, 0)
θ_N

The drawing shows an edge view of a leaf spring, attached at its left end to a fixed support. It is assumed that the spring leaves the support at an angle of θ_0 Most often θ_0 will be 90º.

A force F, at an angle of ϕ with the horizontal, is applied at the end of the spring and a bending moment Q_N is also applied at the end. This is the general case, applicable to springs mounted with fixed attachments. If the spring had hinged end attachments, Q_N would be zero and the line of action of force F, would have to pass through the fixed support point.

The inset shows a greatly magnified detail of one of the finite elements. In that drawing the radius of curvature r_n is shown greatly reduced, which allows the curvature and the trigonometric relationships to be visualized. In the real world, the opposing faces of the element would be nearly parallel, normally differing in direction by only a few tenths of a degree.

For each element, its radius of curvature may be computed by dividing the spring stiffness, E_1I by the bending moment at the element's location. This moment consists of two parts, first Q_N which appears at all points along the spring, plus a variable (negative) moment arising from the end force F, multiplied by its perpendicular distance d_n from element, n. That is $Q_n = Q_N - F\ d_n$

d_n may be computed by means of two triangles, shown in blue on the drawing.
$d_n = (x_N - x_n) \sin\phi + (y_n - y_N) \cos\phi$

We now proceed to build up the spring shape, element by element as follows:

Starting with element 'n' the normal to its first (leftmost) face is at angle θ_n . Given 's' as the length of the elements, and with angles given in radians, the change in angle from the first to the second face $\delta\theta_n$ is simply $-s/r_n$, and we can compute r_n from the spring bending moment at element 'n'. As shown above, $r_n = E_1I / (Q_N - F\ d_n)$, where $d_n = (x_N - x_n) \sin\phi + (y_n - y_N) \cos\phi$. Adding $\delta\theta_n$ to θ_n we get the angle of the second (rightmost) face of element n. $\theta_{n+1} = \theta_n + \delta\theta_n$

Now that the angle of the second face is determined, we can set out to compute its coordinates. The change in the x coordinate in going from the first to the second face is $r_n (\sin\theta_{n+1} - \sin\theta_n)$ and the change in the y coordinate is $r_n (\cos\theta_{n+1} - \cos\theta_n)$ Adding these to the coordinates of the starting face (x_n, y_n) gives us the coordinates of the second face (x_{n+1}, y_{n+1}).

Now we can continue the process by using these ending coordinates and angle to define the starting face for the next element. This process is repeated for each of the elements until the end of the spring is reached, having traversed all N elements.

To start off the process, we know that the first element, element 0, has the known starting angle of θ_0 and its starting coordinates were assumed to be (0,0).