Analysis of a leaf spring under large deflection

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The spring to be analyzed consists of a thin, flat, rectangular sheet of spring steel, bent into a strongly curved "C" shape. Force to maintain the curved shape is applied to the opposing narrow edges of the spring through flexible hinges.

The goal is to examine the force/deflection behavior of the spring. Also the maximum stresses in the spring will be determined.

Spring description:

Width, W = 3.5"
Length, L = 12"
Thickness, T = 0.018"

Elastic Modulus, E = 29 x 10^6 psi (Estimated for steel.)[1]

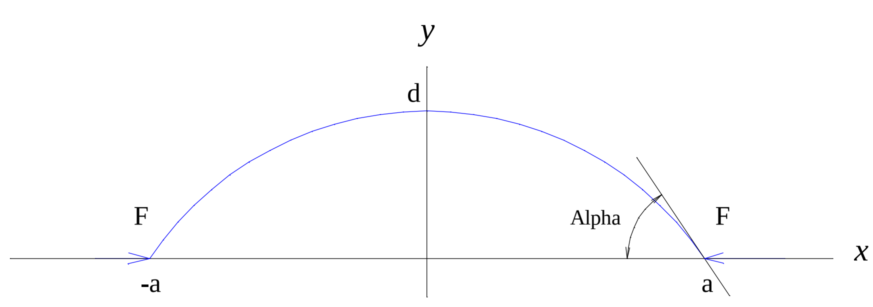
Breaking tensile strength = 180 x 10^3 psi (Estimated for typical spring steel)

Cross-section geometry:

The geometric moment of inertia, I of the spring cross section is given by
$I = T^3 W / 12 = 1.701 \times 10^{-6}\ in^4$

The section modulus, Z of the spring cross section is given by
$Z = T^2 W / 6 = 1.890 \times 10^{-4}\ in^3$

The spring is viewed edge-on to its long dimension and initially lies along the x-axis, (y = 0) from x = –L/2 to +L/2



Force = F pounds is applied along the x-axis, to the narrow ends of the bent spring.

The center of the spring bows up to height, y = d inches (at x = 0).

The ends of the stressed spring are at +/- a inches (distance between ends = 2a)

The bent spring, by definition, takes the shape of an "Elastica" curve.

The Elastica is actually a set of curves, each of which corresponds to a particular amount of bending. The degree of "bend" can be related to Alpha, the angle the ends of the spring make with the x axis. For ease of computation a new variable k, which depends on Alpha, is used to define the degree of bend, where $k = \sin(\text{Alpha} / 2)$.

With no bending the elastica degenerates to the straight line, $y = 0$ from $x = -L/2$ to $+L/2$, and Alpha = 0, $k = 0$

When the spring has been bent to the point that the ends are parallel, then Alpha = 90^o, $k = 0.707...$

As the bending force is increased to bring the ends toward each other, Alpha exceeds 90^o, and k also increases; $0.707... < k < 1$.

The analysis of the bent spring shape is rather complicated. In Reddick & Miller[2] the hard work is all done and it only remained to put their results into the form we need.

The compressed length of the spring, bent by the amount k is $2a = L\,(2\,E(k) / K(k) - 1)$ inches.

where L is the original length of the spring, K(k) is the elliptic integral function of the first kind and E(k) is the elliptic integral function of the second kind.

Values for E(k) and K(k) can be found in mathematical tables[3], though they're not yet on most pocket calculators. There are also some simple computer algorithms which will derive them very accurately.

The force to compress the spring to the same amount, k, is $F = E\,I\,K(k)^2 / 36$ pounds where E is the elastic modulus[1], I is the section moment of inertia and K(k) is the elliptic integral, as above.

By choosing a series of values for parameter k, the corresponding Length / Force data pairs can be computed and plotted. An Excel spreadsheet program was used to tabulate and graph the data for two springs, one of which was subjected to experimental confirmation. See file Spring2.psd

To find the maximum stress in the spring at a given degree of bend, we first need to calculate the depth of bow, at the center of the spring.

Bow depth $d = 12\,k / K(k)$

From that we can compute the maximum stress in the spring bent to degree k
If we ignore longitudinal stresses (< 120 psi) the max stress is given by $F\,d / Z$ psi,
where F is the force, d = the bow depth and Z is the section modulus.

It is interesting to note that 180,000 psi, a not uncommon breaking stress for spring steel, is reached just as the ends of the spring are bent almost to touch. With the ends bent parallel, the max stress was computed at 114,000 psi, and in the STM-8 about 123,000.

Accuracy:

A major error possibility is in my estimate of 29 million psi for the elastic modulus[1] of the spring material. It is probably accurate within roughly 1 million psi, so the data could be in error by a corresponding amount. For any given spring, it may be possible to refine that estimate empirically. Also, the calculations are dependent on the cube of the assumed thickness of the blade. Small deviations from that number could cause noticeable errors. It is also assumed that the spring starts out nearly flat. If it exhibits an initial "set", measurements made on the spring will not agree well with this analysis.

Application:

The formula was applied to spring with dimensions as given in the description of the STM-8 vertical seismograph. From the description of that instrument, a point was found on the curve which approximates the operating point of the STM-8 spring. The tangent line to the curve at that operating point was drawn.

Testing:

The formula was tested by making measurements on a piece of steel shim stock, 12” long x 0.5” wide x 0.017” thick and comparing the values obtained with the theoretical values. The measured force-length curve was satisfyingly similar to the computed values. Nevertheless there is a modest difference in slope between the measured and calculated curves, which has no obvious cause. Errors in Elastic Modulus assumptions[1] or dimensional errors would result in the curve baing displaced to the side, and would not primarily affect the slope of the curve. Any suggestons as to possible causes for this error would be appreciated by the author.

I would particularly like to thank Bob Barns for his measurements on these willful and occasionally airborne leaf springs. His efforts have provided the bridge from an otherwise interesting theory to an actual working tool.

Notes:

1. I have recently discovered an error in the above assumptions which will make a modest (9%) difference in the results. The analysis was done, based on the formula for the bending of a beam, which assumes that the height (thickness) and width of the beam cross-section are of the same order of magnitude. As the beam bends, the volume on the inside of the curve tries to become less (shorter) and the volume on the outside of the curve tries to become greater (longer). Since solids are very resistant to volume compression, the tendency is for the width of the beam on the inside of the curve to increase slightly, and on the outside of the curve, to similarly decrease. If you want to observe this effect, try bending a large rubber eraser.

   In our leaf spring which has its width so much greater than its thickness, the material on the inside and outside of the curve cannot expand and contract sideways, except in very narrow regions close to the edges of the spring. Because of this, when bending, such a spring appears to be slightly stiffer than we would expect; its *apparent* modulus of elasticity, E, will appear to be somewhat larger than the true value. The approximate correction factor is $1/(1 - \sigma^2)$ where σ is Poisson's ratio. For carbon steel σ is measured to be about 0.29, so the effective value of E should be raised by a factor of roughly 1.092. This suggests that we should be using something closer to 32 million psi for E.

2. The Elastica curve, p. 138 of Reddick & Miller
   “Advanced Mathematics for Engineers” 2nd edition
   John Wiley & Sons, New York, 1938

3. C.R.C. “Standard Mathematical Tables” 11th edition
   The Chemical Rubber Publishing Co., Cleveland, 1957

STM-8 Spring Analysis

α deg.	k = Sin(α/2)	K(k)	E(k)	Force = F Pounds	Force = F Newtons	Gap Opening = 2a Inches	Gap Opening = 2a Cm.	Bow Depth = d Inches	Max stress psi
0	0.00000	1.5708	1.5708	3.38	15.04	12.00	30.48	0.00	0
2	0.01745	1.5709	1.5707	3.38	15.04	12.00	30.47	0.13	2,385
4	0.03490	1.5713	1.5703	3.38	15.05	11.98	30.44	0.27	4,771
6	0.05234	1.5719	1.5697	3.39	15.06	11.97	30.39	0.40	7,157
8	0.06976	1.5727	1.5689	3.39	15.08	11.94	30.33	0.53	9,544
10	0.08716	1.5738	1.5678	3.39	15.10	11.91	30.25	0.66	11,933
12	0.10453	1.5751	1.5665	3.40	15.12	11.87	30.15	0.80	14,324
14	0.12187	1.5767	1.5649	3.41	15.15	11.82	30.02	0.93	16,717
16	0.13917	1.5785	1.5632	3.41	15.19	11.77	29.89	1.06	19,113
18	0.15643	1.5805	1.5611	3.42	15.23	11.71	29.73	1.19	21,510
20	0.17365	1.5828	1.5589	3.43	15.27	11.64	29.56	1.32	23,912
22	0.19081	1.5854	1.5564	3.44	15.32	11.56	29.36	1.44	26,318
24	0.20791	1.5882	1.5537	3.46	15.37	11.48	29.16	1.57	28,728
26	0.22495	1.5913	1.5507	3.47	15.43	11.39	28.92	1.70	31,143
28	0.24192	1.5946	1.5476	3.48	15.50	11.29	28.68	1.82	33,562
30	0.25882	1.5981	1.5442	3.50	15.57	11.19	28.42	1.94	35,985
32	0.27564	1.6020	1.5405	3.52	15.64	11.08	28.14	2.06	38,417
34	0.29237	1.6061	1.5367	3.53	15.72	10.96	27.85	2.18	40,853
36	0.30902	1.6105	1.5326	3.55	15.81	10.84	27.53	2.30	43,297
38	0.32557	1.6151	1.5283	3.57	15.90	10.71	27.20	2.42	45,747
40	0.34202	1.6200	1.5238	3.60	16.00	10.57	26.86	2.53	48,204
42	0.35837	1.6252	1.5191	3.62	16.10	10.43	26.50	2.65	50,671
44	0.37461	1.6307	1.5141	3.64	16.21	10.28	26.12	2.76	53,146
46	0.39073	1.6365	1.5090	3.67	16.32	10.13	25.73	2.87	55,631
48	0.40674	1.6426	1.5037	3.70	16.45	9.97	25.33	2.97	58,125
50	0.42262	1.6490	1.4981	3.73	16.57	9.80	24.90	3.08	60,630
52	0.43837	1.6557	1.4924	3.76	16.71	9.63	24.47	3.18	63,146
54	0.45399	1.6627	1.4864	3.79	16.85	9.46	24.02	3.28	65,672
56	0.46947	1.6701	1.4803	3.82	17.00	9.27	23.55	3.37	68,214
58	0.48481	1.6777	1.4740	3.86	17.16	9.09	23.08	3.47	70,763
60	0.50000	1.6858	1.4675	3.89	17.32	8.89	22.59	3.56	73,332
62	0.51504	1.6941	1.4608	3.93	17.49	8.69	22.09	3.65	75,910
64	0.52992	1.7028	1.4539	3.97	17.67	8.49	21.57	3.73	78,504

Notes

Spring made of steel

L = 12.000 in

W = 3.500 in

T = 0.018 in

Modulus of elasticity (steel)

E = 2.9E+07 lb/in^2

Section Moment of inertia

$I = T^3 W / 12 = 1.701E-06 \ in^4$

Section Modulus

$Z = T^2 W / 6 = 1.890E-04 \ in^3$

K(k) is the complete Elliptic Integral of the first kind.

E(k) is the complete Elliptic Integral of the second kind.

E(k) and K(k) values obtained from "C.R.C. Standard Math Tables", 11th ed.

Parameter α is the angle made by each end of the spring with the original plane of the spring.

Force = $(E I K(k)^2 / 36)$ lb.

Gap Length = L (2 E(k) / K(k) - 1) in.

Bow Depth = L k / K(k) in.

Max Stress = F d / Z psi

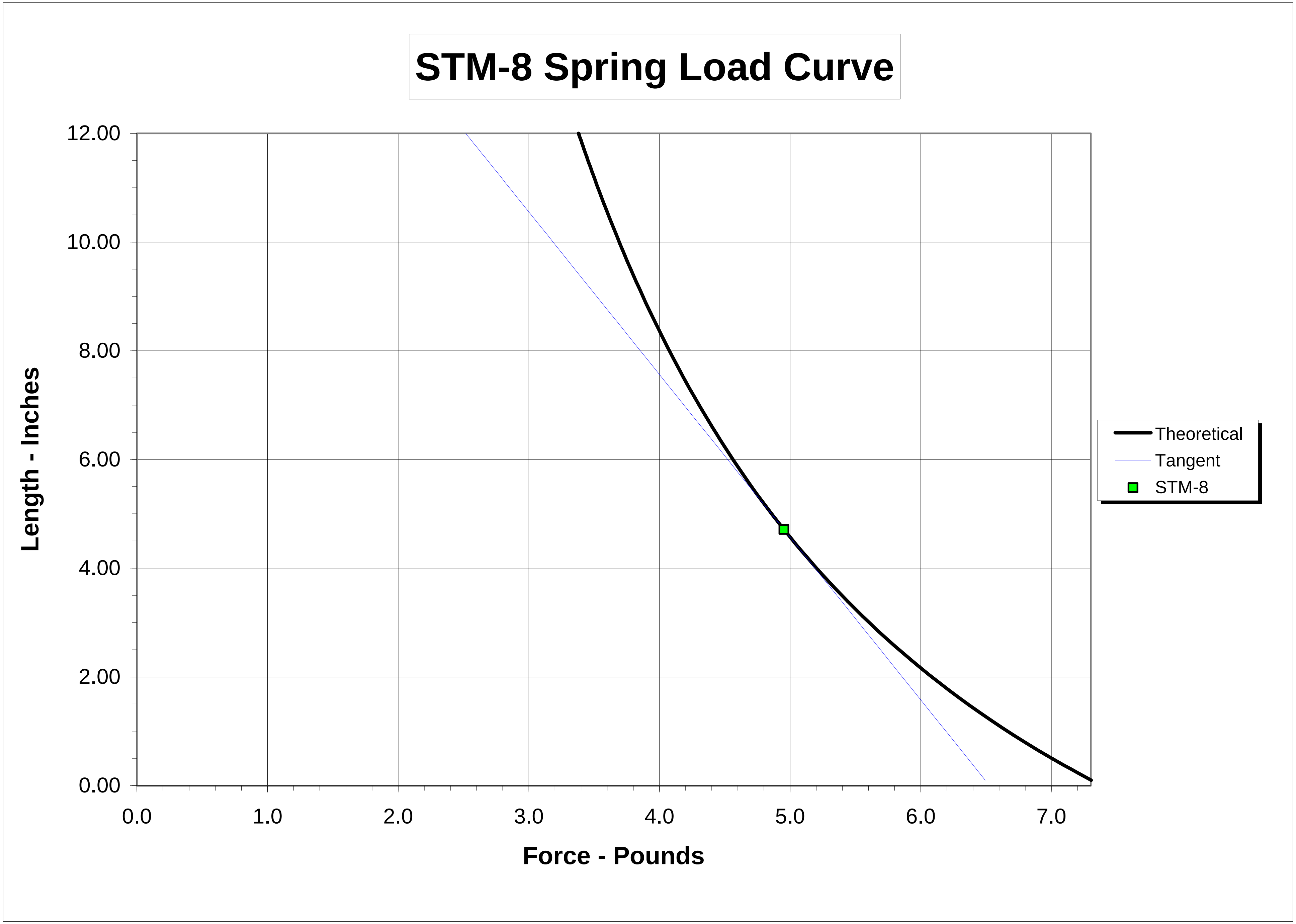
STM-8 Spring Analysis

α deg.	k = Sin(α/2)	K(k)	E(k)	Force = F Pounds	Force = F Newtons	Gap Opening = 2a Inches	Gap Opening = 2a Cm.	Bow Depth = d Inches	Max stress psi	Notes
66	0.54464	1.7119	1.4469	4.02	17.86	8.28	21.04	3.82	81,116	
68	0.55919	1.7214	1.4397	4.06	18.06	8.07	20.50	3.90	83,746	
70	0.57358	1.7312	1.4323	4.11	18.27	7.86	19.95	3.98	86,389	
72	0.58779	1.7415	1.4248	4.16	18.49	7.64	19.39	4.05	89,056	
74	0.60182	1.7522	1.4171	4.21	18.71	7.41	18.82	4.12	91,742	
76	0.61566	1.7633	1.4092	4.26	18.95	7.18	18.24	4.19	94,447	
78	0.62932	1.7748	1.4013	4.32	19.20	6.95	17.65	4.26	97,172	
80	0.64279	1.7868	1.3931	4.37	19.46	6.71	17.05	4.32	99,922	
82	0.65606	1.7992	1.3849	4.44	19.73	6.47	16.44	4.38	102,693	
84	0.66913	1.8122	1.3765	4.50	20.02	6.23	15.82	4.43	105,496	
86	0.68200	1.8256	1.3680	4.57	20.31	5.98	15.20	4.48	108,320	
88	0.69466	1.8396	1.3594	4.64	20.63	5.74	14.57	4.53	111,177	
90	0.70711	1.8541	1.3506	4.71	20.95	5.48	13.93	4.58	114,061	--- *Sides bent parallel*
92	0.71934	1.8691	1.3418	4.79	21.29	5.23	13.28	4.62	116,973	
94	0.73135	1.8848	1.3329	4.87	21.65	4.97	12.63	4.66	119,926	
96	0.74314	1.9011	1.3238	4.95	22.03	4.71	11.97	4.69	122,913	--- *STM-8*
98	0.75471	1.9180	1.3147	5.04	22.42	4.45	11.31	4.72	125,935	*operating point*
100	0.76604	1.9356	1.3055	5.13	22.84	4.19	10.64	4.75	129,000	
102	0.77715	1.9539	1.2963	5.23	23.27	3.92	9.96	4.77	132,106	
104	0.78801	1.9729	1.2870	5.33	23.72	3.66	9.29	4.79	135,256	
106	0.79864	1.9927	1.2776	5.44	24.20	3.39	8.60	4.81	138,455	
108	0.80902	2.0133	1.2681	5.55	24.71	3.12	7.92	4.82	141,705	
110	0.81915	2.0347	1.2587	5.67	25.23	2.85	7.23	4.83	145,005	
112	0.82904	2.0571	1.2492	5.80	25.79	2.57	6.54	4.84	148,371	
114	0.83867	2.0804	1.2397	5.93	26.38	2.30	5.85	4.84	151,795	
116	0.84805	2.1047	1.2301	6.07	27.00	2.03	5.15	4.84	155,285	
118	0.85717	2.1300	1.2206	6.22	27.65	1.75	4.45	4.83	158,842	
120	0.86603	2.1565	1.2111	6.37	28.35	1.48	3.76	4.82	162,480	
122	0.87462	2.1842	1.2015	6.54	29.08	1.20	3.05	4.81	166,200	
124	0.88295	2.2132	1.1920	6.71	29.86	0.93	2.35	4.79	170,010	
126	0.89101	2.2435	1.1826	6.90	30.68	0.65	1.65	4.77	173,911	
128	0.89879	2.2754	1.1732	7.09	31.56	0.37	0.95	4.74	177,925	
130	0.90631	2.3088	1.1638	7.30	32.49	0.10	0.25	4.71	182,046	--- *Ends almost touching*

STM-8 Spring Analysis

α deg.	k = Sin(α/2)	K(k)	E(k)	Force = F Pounds	Force = F Newtons	Gap Opening = 2a Inches	Gap Opening = 2a Cm.	Bow Depth = d Inches	Max stress psi	Notes
132	0.91355	2.3439	1.1545	7.53	33.49	0.00	-0.45			*Ends intersecting.*
134	0.92050	2.3809	1.1453	7.77	34.55	-0.01	-1.16			*Not physically realizable.*
136	0.92718	2.4198	1.1362	8.02	35.69	-0.02	-1.86			
138	0.93358	2.4610	1.1272	8.30	36.92	-0.03	-2.56			
140	0.93969	2.5046	1.1184	8.60	38.23	-0.03	-3.26			
142	0.94552	2.5507	1.1096	8.91	39.66	-0.04	-3.96			
144	0.95106	2.5998	1.1011	9.26	41.20	-0.05	-4.66			
146	0.95630	2.6521	1.0927	9.64	42.87	-0.05	-5.36			
148	0.96126	2.7081	1.0844	10.05	44.70	-0.06	-6.07			
150	0.96593	2.7681	1.0764	10.50	46.70	-0.07	-6.78			
152	0.97030	2.8327	1.0686	11.00	48.91	-0.07	-7.48			
154	0.97437	2.9026	1.0611	11.54	51.35	-0.08	-8.19			
156	0.97815	2.9786	1.0538	12.16	54.08	-0.09	-8.91			
158	0.98163	3.0617	1.0468	12.84	57.14	-0.10	-9.64			
160	0.98481	3.1534	1.0401	13.63	60.61	-0.10	-10.37			
162	0.98769	3.2553	1.0338	14.52	64.59	-0.11	-11.12			
164	0.99027	3.3699	1.0278	15.56	69.22	-0.12	-11.89			Brett Nordgren
166	0.99255	3.5004	1.0223	16.79	74.68	-0.13	-12.68			bnordgren@koryubooks.com
168	0.99452	3.6519	1.0172	18.27	81.29	-0.14	-13.50			19-Feb-99
170	0.99619	3.8317	1.0127	20.12	89.49	-0.14	-14.37			
172	0.99756	4.0530	1.0086	22.51	100.12	-0.15	-15.31			This analysis is largely based on a
174	0.99863	4.3390	1.0053	25.80	114.75	-0.16	-16.36			discussion of the Elastica curve on p. 138
176	0.99939	4.7430	1.0026	30.83	137.12	-0.18	-17.59			of Reddick & Miller - "Advanced
178	0.99985	5.4350	1.0008	40.48	180.05	-0.19	-19.25			Mathematics for Engineers" - 2nd Edition
180	1.00000	inf.	1.0000	inf.	inf.	n/a	n/a			New York, John Wiley & Sons, 1938

STM -8 Spring Analysis



Bending of a Feeler Gauge

α deg	k = Sin(α/2)	K(k)	E(k)	Force - F gms	Gap Opening - 2a in.	Bow Depth - d in.	Max stress psi
0	0.00000	1.5708	1.5708	184.95	12.00	0.00	0
2	0.01745	1.5709	1.5707	184.97	12.00	0.13	2,253
4	0.03490	1.5713	1.5703	185.07	11.98	0.27	4,506
6	0.05234	1.5719	1.5697	185.21	11.97	0.40	6,760
8	0.06976	1.5727	1.5689	185.40	11.94	0.53	9,014
10	0.08716	1.5738	1.5678	185.66	11.91	0.66	11,270
12	0.10453	1.5751	1.5665	185.96	11.87	0.80	13,528
14	0.12187	1.5767	1.5649	186.34	11.82	0.93	15,788
16	0.13917	1.5785	1.5632	186.77	11.77	1.06	18,051
18	0.15643	1.5805	1.5611	187.24	11.71	1.19	20,315
20	0.17365	1.5828	1.5589	187.78	11.64	1.32	22,584
22	0.19081	1.5854	1.5564	188.40	11.56	1.44	24,856
24	0.20791	1.5882	1.5537	189.07	11.48	1.57	27,132
26	0.22495	1.5913	1.5507	189.81	11.39	1.70	29,413
28	0.24192	1.5946	1.5476	190.60	11.29	1.82	31,697
30	0.25882	1.5981	1.5442	191.43	11.19	1.94	33,986
32	0.27564	1.6020	1.5405	192.37	11.08	2.06	36,282
34	0.29237	1.6061	1.5367	193.35	10.96	2.18	38,584
36	0.30902	1.6105	1.5326	194.42	10.84	2.30	40,892
38	0.32557	1.6151	1.5283	195.53	10.71	2.42	43,205
40	0.34202	1.6200	1.5238	196.72	10.57	2.53	45,526
42	0.35837	1.6252	1.5191	197.98	10.43	2.65	47,855
44	0.37461	1.6307	1.5141	199.32	10.28	2.76	50,193
46	0.39073	1.6365	1.5090	200.74	10.13	2.87	52,540
48	0.40674	1.6426	1.5037	202.24	9.97	2.97	54,896
50	0.42262	1.6490	1.4981	203.82	9.80	3.08	57,262
52	0.43837	1.6557	1.4924	205.48	9.63	3.18	59,637
54	0.45399	1.6627	1.4864	207.22	9.46	3.28	62,024
56	0.46947	1.6701	1.4803	209.07	9.27	3.37	64,424
58	0.48481	1.6777	1.4740	210.98	9.09	3.47	66,831
60	0.50000	1.6858	1.4675	213.02	8.89	3.56	69,258

Notes

Spring made of steel

L = 12.000 in
W = 0.500 in
T = 0.017 in

Modulus of elasticity (steel)

E = 2.9E+07 lb/in^2

Section Moment of inertia

2.0471E-07 in^4

Section Modulus

2.4083E-05 in^3

K(k) is the complete Elliptic Integral of the first kind.

E(k) is the complete Elliptic Integral of the second kind.

E(k) and K(k) values obtained from "C.R.C. Standard Math Tables", 11th ed.

Parameter Alpha is the angle made by each end of the spring with the original plane of the spring.

Force = $E\ I\ K(k)^2 / 36$ lb.
Gap Length = $L\ (2\ E(k) / K(k) - 1)$ in.
Bow Depth = $L\ k / K(k)$ in.
Max Stress = $F\ d / Z$ psi

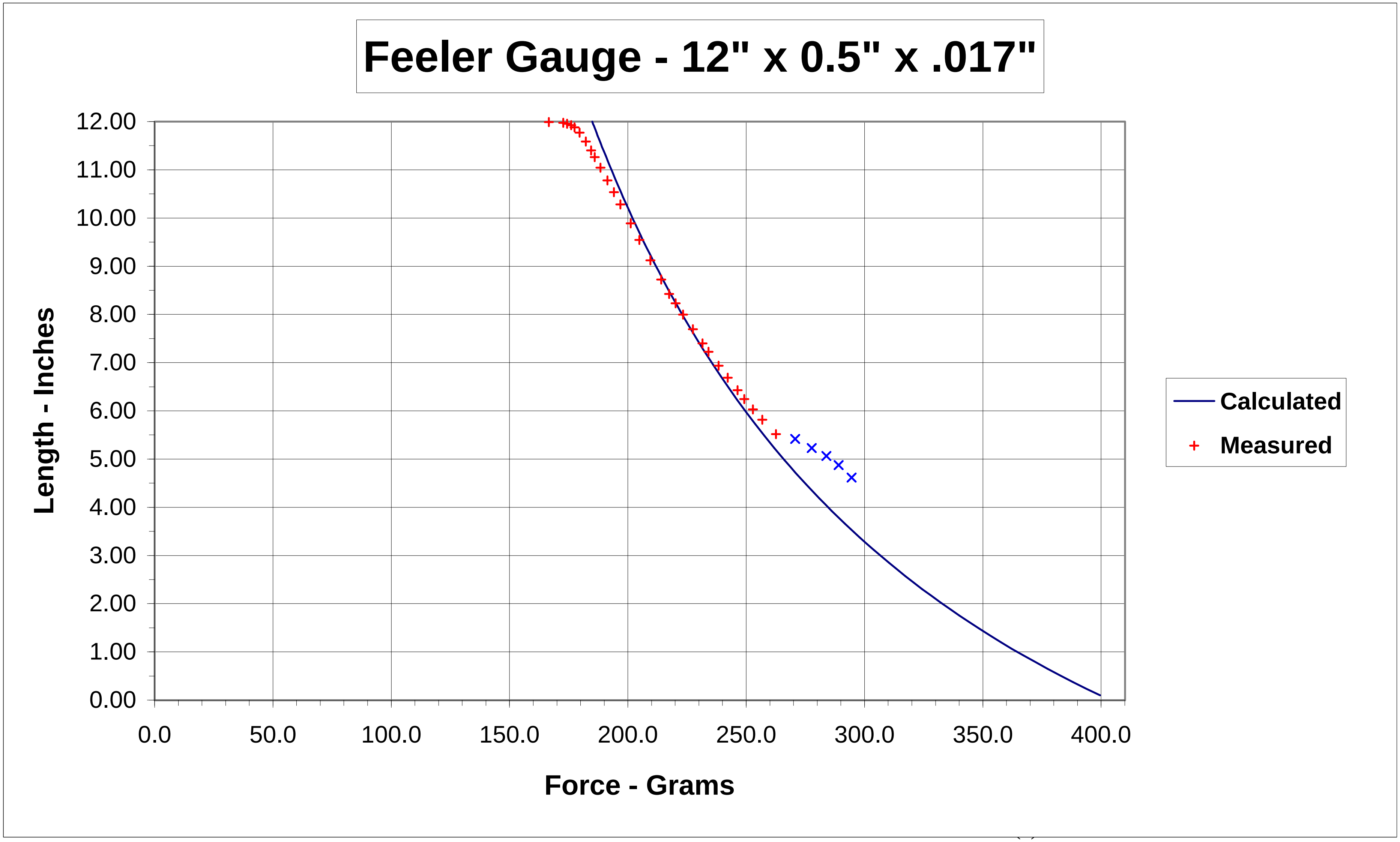
Bending of a Feeler Gauge

α deg	k = Sin(α/2)	K(k)	E(k)	Force - F gms	Gap Opening - 2a in.	Bow Depth - d in.	Max stress psi	Notes
62	0.51504	1.6941	1.4608	215.12	8.69	3.65	71,693	
64	0.52992	1.7028	1.4539	217.34	8.49	3.73	74,143	
66	0.54464	1.7119	1.4469	219.67	8.28	3.82	76,610	
68	0.55919	1.7214	1.4397	222.11	8.07	3.90	79,093	
70	0.57358	1.7312	1.4323	224.65	7.86	3.98	81,589	
72	0.58779	1.7415	1.4248	227.33	7.64	4.05	84,108	
74	0.60182	1.7522	1.4171	230.13	7.41	4.12	86,645	
76	0.61566	1.7633	1.4092	233.06	7.18	4.19	89,200	
78	0.62932	1.7748	1.4013	236.11	6.95	4.26	91,773	
80	0.64279	1.7868	1.3931	239.31	6.71	4.32	94,371	
82	0.65606	1.7992	1.3849	242.64	6.47	4.38	96,988	
84	0.66913	1.8122	1.3765	246.16	6.23	4.43	99,635	
86	0.68200	1.8256	1.3680	249.82	5.98	4.48	102,302	
88	0.69466	1.8396	1.3594	253.66	5.74	4.53	105,000	
90	0.70711	1.8541	1.3506	257.68	5.48	4.58	107,724	*Sides bent parallel*
92	0.71934	1.8691	1.3418	261.86	5.23	4.62	110,475	
94	0.73135	1.8848	1.3329	266.28	4.97	4.66	113,263	
96	0.74314	1.9011	1.3238	270.91	4.71	4.69	116,084	
98	0.75471	1.9180	1.3147	275.74	4.45	4.72	118,939	
100	0.76604	1.9356	1.3055	280.83	4.19	4.75	121,833	
102	0.77715	1.9539	1.2963	286.16	3.92	4.77	124,767	
104	0.78801	1.9729	1.2870	291.76	3.66	4.79	127,742	
106	0.79864	1.9927	1.2776	297.64	3.39	4.81	130,763	
108	0.80902	2.0133	1.2681	303.83	3.12	4.82	133,833	
110	0.81915	2.0347	1.2587	310.32	2.85	4.83	136,950	
112	0.82904	2.0571	1.2492	317.19	2.57	4.84	140,128	
114	0.83867	2.0804	1.2397	324.42	2.30	4.84	143,362	
116	0.84805	2.1047	1.2301	332.04	2.03	4.84	146,658	
118	0.85717	2.1300	1.2206	340.07	1.75	4.83	150,017	
120	0.86603	2.1565	1.2111	348.58	1.48	4.82	153,453	
122	0.87462	2.1842	1.2015	357.60	1.20	4.81	156,967	

Bending of a Feeler Gauge

α deg	k = Sin(α/2)	K(k)	E(k)	Force - F gms	Gap Opening - 2a in.	Bow Depth - d in.	Max stress psi	Notes
124	0.88295	2.2132	1.1920	367.16	0.93	4.79	160,565	
126	0.89101	2.2435	1.1826	377.28	0.65	4.77	164,249	
128	0.89879	2.2754	1.1732	388.08	0.37	4.74	168,040	
130	0.90631	2.3088	1.1638	399.56	0.10	4.71	171,932	*Ends almost touching*
132	0.91355	2.3439	1.1545	411.80	-0.18			*Ends intersecting.*
134	0.92050	2.3809	1.1453	424.90	-0.46			*Not physically realizable.*
136	0.92718	2.4198	1.1362	438.90	-0.73			
138	0.93358	2.4610	1.1272	453.97	-1.01			
140	0.93969	2.5046	1.1184	470.20	-1.28			
142	0.94552	2.5507	1.1096	487.67	-1.56			
144	0.95106	2.5998	1.1011	506.63	-1.84			
146	0.95630	2.6521	1.0927	527.22	-2.11			
148	0.96126	2.7081	1.0844	549.72	-2.39			
150	0.96593	2.7681	1.0764	574.34	-2.67			
152	0.97030	2.8327	1.0686	601.46	-2.95			
154	0.97437	2.9026	1.0611	631.51	-3.23			
156	0.97815	2.9786	1.0538	665.02	-3.51			
158	0.98163	3.0617	1.0468	702.64	-3.79			
160	0.98481	3.1534	1.0401	745.36	-4.08			
162	0.98769	3.2553	1.0338	794.31	-4.38			
164	0.99027	3.3699	1.0278	851.22	-4.68			Brett Nordgren
166	0.99255	3.5004	1.0223	918.42	-4.99			bnordgren@koryubooks.com
168	0.99452	3.6519	1.0172	999.65	-5.32			October 16, 1998
170	0.99619	3.8317	1.0127	1100.50	-5.66			
172	0.99756	4.0530	1.0086	1231.29	-6.03			This analysis is largely based on a
174	0.99863	4.3390	1.0053	1411.20	-6.44			discussion of the Elastica curve on p. 138
176	0.99939	4.7430	1.0026	1686.22	-6.93			of Reddick & Miller - "Advanced
178	0.99985	5.4350	1.0008	2214.15	-7.58			Mathematics for Engineers" - 2nd Edition
180	1.00000	inf.	1.0000	inf.	n/a			

Bending of a feeler gauge



Bob Barns' Measurements

166.60	11.992	186.02	11.263	214.03	8.722	242.14	6.688	283.88	5.064
172.65	11.973	188.46	11.043	217.43	8.425	246.34	6.426	289.01	4.873
174.34	11.955	191.32	10.778	220.13	8.231	249.22	6.246	294.55	4.614
176.02	11.926	194.05	10.534	223.29	7.996	252.83	6.031	Force not applied	
177.45	11.881	196.80	10.280	227.45	7.693	256.79	5.815	at ends of spring.	
179.63	11.769	201.16	9.887	231.54	7.401	262.59	5.516		
182.21	11.585	204.86	9.546	234.06	7.227	270.69	5.418		
184.42	11.401	209.55	9.124	238.38	6.934	277.74	5.230		