We may thus proceed in the following manner. We first find, by trial and error, the value of k from the implicit relation (8), in which $c = 2\sqrt{T/\rho g}$ and ϕ_1 is given by (7). Then the definition (5) of k yields y_0, and finally the equations (6) and (4) give us x and y in terms of the parameter ϕ. In particular, the maximum rise h between the plates is given by (3).

In connection with the relation (3) between the capillary rises outside and between the plates, we noted that $h > h_0$ provided $y_0 \neq 0$. Now if y_0 were zero, we should have $k = 1$ by (5). Then $K(k)$ would become infinite (see Art. 30, Problem 1), while $F(k, \phi_1)$, $E(k)$, and $E(k, \phi_1)$ all remain finite, so that by (8) we should have a infinite, contrary to supposition. Hence the liquid will always rise to a higher level between the plates than it will outside them.

33. The elastica. As another application of elliptic integrals, we consider the *elastica*, the curve assumed by a uniform elastic spring, originally straight, the ends of which are subjected to two equal and opposite compressive forces (Fig. 28). Let the x-axis be taken through

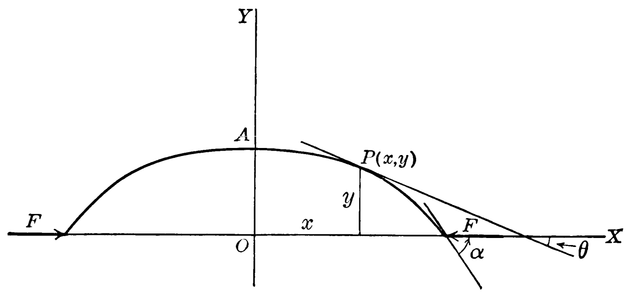

FIG. 28

the ends of the spring and the y-axis through its midpoint. Let the magnitude of the force applied at each end, along the x-axis, be F (lb.), and let the tangent at any point $P(x, y)$ of the curve make an angle θ with the negative direction of the x-axis, x and y being measured in inches. Then the bending moment equation [Chapter I, Art. 7(*f*)] gives us

$$\frac{EI}{R} = M = -Fy, \tag{1}$$

where E (lb./in.^{2}) is the modulus of elasticity, I (in.^{4}) is the moment of inertia of the cross-sectional area of the spring with respect to a line perpendicular to the xy-plane and through the neutral section, and R (in.) is the radius of curvature at the point P.

If s (in.) is the arc length AP, we have

$$\frac{1}{R} = -\frac{d\theta}{ds} = -\frac{dy}{ds}\frac{d\theta}{dy} = \sin\theta\,\frac{d\theta}{dy};$$

hence (1) may be written

$$EI \sin\theta\, d\theta = -Fy\, dy,$$

or, letting $c^2 = EI/F$,

$$y\, dy = -c^2 \sin\theta\, d\theta. \tag{2}$$

Integration yields

$$\frac{y^2}{2} = c^2 \cos\theta + c_1.$$

Denoting by α the value of θ at the end of the spring where $y = 0$, we find $c_1 = -c^2 \cos\alpha$, and therefore

$$y = \sqrt{2}\, c \sqrt{\cos\theta - \cos\alpha}. \tag{3}$$

Replacing $\sin\theta$ by $-dy/ds$ in (2), we obtain

$$ds = c^2 \frac{d\theta}{y} = \frac{c}{\sqrt{2}} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Substituting $\cos\theta = 1 - 2\sin^2(\theta/2)$, $\cos\alpha = 1 - 2\sin^2(\alpha/2)$, this becomes

$$ds = \frac{c}{2} \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}.$$

Letting $k = \sin(\alpha/2)$ and $\sin(\theta/2) = k \sin\phi$, whence

$$\cos\frac{\theta}{2}\frac{d\theta}{2} = k\cos\phi\, d\phi$$

and

$$d\theta = \frac{2k\cos\phi\, d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

we have

$$s = c\int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = cF(k, \phi),$$

and (3) reduces to

$$y = 2c\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}} = 2ck\cos\phi.$$

To find x in terms of ϕ, we have

$$dx = \cos\theta \cdot ds = (1 - 2k^2 \sin^2 \phi) \frac{c\, d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

$$x = c\int_0^\phi \frac{2(1 - k^2 \sin^2 \phi) - 1}{\sqrt{1 - k^2 \sin^2 \phi}}\, d\phi = c[2E(k, \phi) - F(k, \phi)].$$

Summarizing, we have for the parametric equations of the elastica,

$$x = c[2E(k, \phi) - F(k, \phi)], \tag{4}$$

$$y = 2ck \cos \phi, \tag{5}$$

and for the arc length AP,

$$s = cF(k, \phi), \tag{6}$$

where

$$c = \sqrt{\frac{EI}{F}}, \qquad k = \sin\frac{\alpha}{2}, \qquad \phi = \sin^{-1}\frac{\sin(\theta/2)}{\sin(\alpha/2)}.$$

If $2a$, b, and L, measured in inches, are, respectively, the distance between the ends of the spring, the maximum deflection, and the length of the spring, we have, since $\phi = \pi/2$ when $\theta = \alpha$ and $\phi = 0$ when $\theta = 0$,

$$a = c[2E(k) - K(k)], \tag{4'}$$

$$b = 2ck, \tag{5'}$$

$$L = 2cK(k). \tag{6'}$$

Equation (6′) is also of particular interest in that it indicates a critical value for L. For, the function $K(\sin \alpha/2)$ must be greater than $\pi/2$ (see Art. 30, Problem 1), and therefore L must be greater than πc to produce the supposed bent form; if $L < \pi c$, we have simple compression without bending. An equivalent statement is that, for a given value of L, the number $c = \sqrt{EI/F}$ must be sufficiently small if bending is to result; this may be brought about by decreasing E or I, or more simply by increasing the force F.

34. The swinging cord. Our next problem is the determination of the curve assumed by a skipping rope (Fig. 29). Consider a uniform cord or chain rotating about a horizontal axis to which the ends are fixed at two points a distance $2a$ (ft.) apart. Let ω (rad./sec.) denote the constant angular velocity of rotation, which we suppose so large that centrifugal force predominates over the gravitational force due to the small weight of the cord; we accordingly consider only the former force. Let w (lb./ft.) be the weight per foot of cord, and let t (pdl.)