II. Complex Variables 101

Introduction

A typical application of complex variables comes about when we attempt to solve the general quadratic equation $s = (ax^2 + bx + 1)$ or its equivalent, $s - (ax^2 + bx + 1) = 0$

Using the quadratic formula we get $s = \frac{-b}{2a} \frac{\pm\sqrt{b^2 - 4a)}}{2a}$

If $(b^2 - 4a)$ is zero or positive the above expression gives two (real) roots. If it is negative, we must make use of the imaginary constant j , defined by $j^2 \equiv -1$

Note: Although mathematicians generally use the symbol i for this constant, in applied mathematics it is usually called j, probably to avoid confusion with the electrical current variable, i.

The roots of the equation are the two complex numbers $s = -\frac{b}{2a} \pm j \frac{\sqrt{(4a - b^2)}}{2a}$

In general, polynomial equations require the use of complex variables for their solutions.

Complex Variables

Complex numbers are two-dimensional. Rather than describing points on a line, they describe points in a plane. And that is their particular usefulness, to describe and manipulate quantities that are essentially two-dimensional in nature. Moreover, complex variables have nearly the same properties as real variables. In particular, algebraic manipulation of complex variables is virtually identical. We can factor and evaluate polynomials of complex variables in the usual way.

Any complex number s, may be written in the form $s = x + j y$ Then s can be plotted in the complex plane, with its real part being the x coordinate and its “imaginary” part, the part multiplied by j, being the y coordinate. The x axis of the complex plane has thus come to be called the *real axis* and the y axis, the *imaginary axis*. Typically, when problems are solved using complex variables, they are resolved into their component values only near the end of the process.

Transfer Functions

When we work with linear feedback systems, we are mostly dealing with the mathematics of transfer functions. A transfer function describes the sinusoidal output of a system element as a function of its sinusoidal input. Here we define a system element as a single block in the system block diagram, a group of such blocks, or even the entire system. Since we are dealing with essentially linear systems, the output of a system element will be a sine wave of the same frequency, (but probably differing amplitude and phase) as its input.

The complete transfer function of a system element will describe its
(1) *gain* = (output amplitude ÷ input amplitude), and

(2) *phase* = (output phase angle – input phase angle)
for any frequency of interest.

It is often useful to express a complex variable in polar rather than rectangular coordinates. Its *magnitude* (or amplitude) is the distance of the point s from the origin, and its *argument* (or angle or phase angle) is the polar angle of the line drawn from the origin to s. An important use of this form arises when we want to use a complex variable to describe the gain and phase of a transfer function. In general terms we can express that description in terms of complex variables as follows:

Assuming an input s, the output of the subsystem will be some function F(s). If s is a complex variable which somehow represents a sine wave of amplitude 1 and phase angle 0, then from (1) the gain of the transfer function is just the magnitude of F(s) and from (2) the phase angle of the transfer function is just the argument of F(s).

Through the magic of complex variables, it turns out that $s = j\,\omega$ defines just such a sine wave, having unit amplitude and zero phase, where ω is its frequency expressed in radians per second.

This results in the important and much-used fact:
The transfer function of a subsystem which has output F(s) *for input* s*, is given by* $F(j\omega)$

Complex polynomials

The transfer functions of subsystems which are linear (or at least linear enough) can be described by products and quotients of complex polynomials. It has been proven that any such polynomial expressions can always be reduced to the form

$$(3)\quad F(s) = C\,\frac{(D_1s^2 + E_1s + 1)(....)\,(s + F_1)(s + F_2)(....)}{(G_1s^2 + H_1s + 1)(....)\,(s + K_1)(s+ K_2)(....)}$$

where there are 0 or more quadratic terms and 0 or more first order terms, each, in the numerator and denominator, and where C, D_n, E_n, F_n, etc. are constants (including zero). That is to say, it should be possible to factor any quotient of complex polynomials (often with some effort) into such a form.

We should note that the quadratic terms correspond to resonant subsystems, such as electrical L-C circuits, and mechanical spring-mass systems. The first order terms correspond to R-C or R-L circuits. The integral of a function is created by multiplying it by 1/s, and similarly, the derivative is formed by multiplying it by s.

The process by which the complex transfer functions, which describe a particular circuit or mechanical system, are created in the first place, is well beyond the scope of this outline, but textbooks on feedback system analysis frequently contain tables listing the complex functions which correspond to most common system elements.

The values (location) of the roots of the numerator and denominator polynomials are of particular interest when dealing with transfer functions. Roots of the numerator polynomial are called "zeros" and are values of s (points in the plane) where the numerator (and hence the quotient) = 0. "Poles" are values of s where the denominator = 0, which are points where the value of the quotient becomes infinite.

When the polynomial is factored into the form of (3) (i.e. when its poles and zeroes are apparent) we may sketch the magnitude of the transfer function (vs frequency) simply by inspection. Often that is all that is needed to understand and correct a loop problem. In the pre-MathCad days, if you weren't trying to do an exceptionally thorough analysis, such a sketch was usually sufficient to complete the analysis.

It is worth noting that all even powers, of j reduce to either +1 or –1. $j^2 = -1$, $j^4 = 1$, etc. Odd powers of j reduce to either +j or –j. $j^1 = j$, $j^3 = j \times j^2 = -j$, $j^5 = j \times j^4 = j$, etc. Making use of this, we can see that any polynomial in j,

(4) $A_n j^n + A_{n-1} j^{n-1} \ldots A_1 j + A_0$ reduces to an expression of the form x + jy

More on using the polar form

The magnitude and argument of a complex variable s can easily be determined by considering that s is a point in the complex plane.

We saw that the magnitude of a complex variable s = x + jy is the distance of s from the origin of the complex plane. Pythagoras tells us, therefore, that the magnitude of x + jy,

written $|x + jy| = \sqrt{x^2 + y^2}$

The argument of s, we said, was defined as the angle of the vector from the origin to s, and so $\theta \equiv \arg(x + jy) = \tan^{-1}(y / x)$, where $\tan^{-1}$ represents the arctangent function.

A particularly useful characteristic of complex variables is that

The magnitude of a product is the product of the magnitudes, written out:

$|A(s)B(s)| = |A(s)|\,|B(s)|$ and by extension $|A(s) \div B(s)| = |A(s)| \div |B(s)|$

This allows us to compute the magnitude of a large complex expression by multiplying together the magnitudes of its individual terms.

Another useful relationship is:

The argument of a product is the sum of the arguments.

or Arg(A(s)B(s)) = Arg(B(s)) + Arg(B(s))
and similarly, Arg(A(s) ÷ B(s)) = Arg(A(s)) – Arg(B(s))

These relationships are particularly useful when evaluating the gain and phase of cascaded system elements. The total gain will be the product of the individual gains of the elements, and the total phase is the sum of the individual phase angles of the elements.

There are no such simple relations which apply to the sums of complex numbers. Things go easiest, as indicated above, when a large expression can be factored into the product of its several terms. A practical implication of this can be seen in the STM-8 loop. The fact that the feedback path is the *sum* of three branches, proportional, derivative, and integral, rather than a being a single path, results in a more complicated expression for B than we might otherwise obtain.

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